Information Theory & Source Coding

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Information Theory

Information theory deals with representation and the transfer of information.

There are two fundamentally different ways to transmit **messages**: via **discrete** signals and via continuous signals..... For example, the letters of the English alphabet are commonly thought of as **discrete** signals.

Information sources

Definition:

The set of source symbols is called the **source alphabet**, and the elements of the set are called the **symbols or letters.**

The number of possible answers ' r ' should be linked to "information."

"Information" should be additive in some sense.

We define the following measure of information:

$$\tilde{I}(U) \triangleq \log_b r,$$

Where ' r ' is the number of all possible outcome so far an do m message U.

Using this definition we can confirm that it has the wanted property of additivity:

$$\tilde{I}(U_1, U_2, \ldots, U_n) = \log_b r^n = n \cdot \log_b r = n\tilde{I}(U).$$

The basis 'b' of the logarithm b is only a change of units without actually changing the amount of information it describes.

Classification of information sources

1. Discrete memory less.
2. Memory.

Discrete memory less source (DMS) can be characterized by "the list of the symbols, the probability assignment to these symbols, and the specification of the rate of generating these symbols by the source".

1. Information should be proportion to the uncertainty of an outcome.
2. Information contained in independent outcome should add.

Scope of Information Theory

1. Determine the irreducible limit below which a signal cannot be compressed.
2. Deduce the ultimate transmission rate for reliable communication over a noisy channel.
3. Define Channel Capacity - the intrinsic ability of a channel to convey information.

The basic setup in Information Theory has:

– a source,

– a channel and

– destination.

The output from source is conveyed through the channel and received at the destination. The source is a random variable S

which takes symbols from a finite alphabet i.e.,

$$S = \{s0, s1, s2, \cdots, sk-1\}$$

With probabilities

$P(S = sk) = pk$ where $k = 0, 1, 2, \cdots, k - 1$

and

$k-1, Xk=0, pk = 1$

The following assumptions are made about the source

1. Source generates symbols that are statistically independent.
2. Source is memory less i.e., the choice of present symbol does not depend on the previous choices.

Properties of Information

1. Information conveyed by a deterministic event is nothing
2. Information is always positive.
3. Information is never lost.
4. More information is conveyed by a less probable event than a more probable event

Entropy:

The Entropy (H(s)) of a source is defined as the average information generated by a discrete memory less source.

Information content of a symbol:

Let us consider a discrete memory less source (DMS) denoted by X and having the alphabet $\{U_1, U_2, U_3, \ldots\ldots U_m\}$. The information content of the symbol xi, denoted by $I(x_i)$ is defined as

$$I(U) = \log_b \frac{1}{P(u)} = -\log_b P(U)$$

Where P (U) is the probability of occurrence of symbol U

Units of $I(x_i)$:

For two important and one unimportant special cases of b it has been agreed to use the following names for these units:

b =2(log2): bit,

b = e (ln): nat (natural logarithm),

b =10(log10): Hartley.

The conversation of these units to other units is given as

$$\log_2 a = \frac{\ln a}{\ln 2} = \frac{\log a}{\log 2}$$

Uncertainty or Entropy (i.e Average information)

Definition:

In order to get the information content of the symbol, the flow information on the symbol can fluctuate widely because of randomness involved into the section of symbols.

The uncertainty or entropy of a discrete random variable (RV) 'U' is defined as

$$H(U) = E[I(u)] = \sum_{i=1}^{m} P(u)I(u)$$

$$H(U) \triangleq -\sum_{u \in \mathrm{supp}(P_U)} P_U(u) \log_b P_U(u),$$

Where PU (·) denotes the probability mass function (PMF) 2 of the RV U, and where the support of P U is defined as

$$\operatorname{supp}(P_U) \triangleq \{u \in \mathcal{U} : P_U(u) \neq 0\}.$$

We will usually neglect to mention "support" when we sum over PU (u) · $\log_b$ PU (u), i.e., we implicitly assume that we exclude all u

With zero probability PU (u) =0.

Entropy for binary source

It may be noted that for a binary source U which genets independent symbols 0 and 1 with equal probability, the source entropy H (u) is

$$H(u) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1 \text{ b/symbol}$$

Bounds on H (U)

If U has r possible values, then $0 \leq H(U) \leq \log r$,

Where

H(U)=0 if, and only if, PU(u)=1 for some u,

H(U)=log r if, and only if, PU(u)= 1/r ∀ u.

Hence, H(U) ≥ 0.Equalitycanonlybeachievedif −PU(u)log2 PU(u)=0

Proof. Since $0 \leq P_U(u) \leq 1$, we have

$$-P_U(u)\log_2 P_U(u) \begin{cases} = 0 & \text{if } P_U(u) = 1, \\ > 0 & \text{if } 0 < P_U(u) < 1. \end{cases}$$

For all u ∈ supp (PU), i.e., PU (u) =1forall u ∈ supp (PU).

To derive the upper bound we use at rick that is quite common in.

Formation theory: We take the deference and try to show that it must be non positive.

$$
\begin{aligned}
H(U) - \log r &= -\sum_{u \in \mathrm{supp}(P_U)} P_U(u) \log P_U(u) - \log r \\
&= -\sum_{u \in \mathrm{supp}(P_U)} P_U(u) \log P_U(u) - \sum_{u \in \mathrm{supp}(P_U)} P_U(u) \log r \\
&= -\sum_{u \in \mathrm{supp}(P_U)} P_U(u) \log \left(P_U(u) \cdot r \right) \\
&= \sum_{u \in \mathrm{supp}(P_U)} P_U(u) \log \Bigg(\underbrace{\frac{1}{r \cdot P_U(u)}}_{\triangleq \xi} \Bigg) \\
&\leq \sum_{u \in \mathrm{supp}(P_U)} P_U(u) \left(\frac{1}{r \cdot P_U} - 1 \right) \cdot \log e \\
&= \left(\sum_{u \in \mathrm{supp}(P_U)} \frac{1}{r} - \underbrace{\sum_{u \in \mathrm{supp}(P_U)} P_U(u)}_{=1} \right) \cdot \log e \\
&= \left(\frac{1}{r} \sum_{u \in \mathrm{supp}(P_U)} 1 - 1 \right) \log e \\
&\leq \left(\frac{1}{r} \sum_{u \in \mathcal{U}} 1 - 1 \right) \log e \\
&= \left(\frac{1}{r} \cdot r - 1 \right) \log e \\
&= (1 - 1) \log e = 0.
\end{aligned}
$$

Equality can only be achieved if

1. In the IT Inequality $\xi = 1$, i.e., if $1r \cdot PU(u) = 1 \Rightarrow PU(u) = 1r$, for all u;
2. |supp (PU)| = r.

Note that if Condition1 is satisfied, Condition 2 is also satisfied.

Conditional Entropy

Similar to probability of random vectors, there is nothing really new about conditional probabilities given that a particular event Y = y has occurred.

The conditional entropy or conditional uncertainty of the RV X given the event Y = y is defined as

$$H(X|Y=y) \triangleq -\sum_{x \in \text{supp}(P_{X|Y}(\cdot|y))} P_{X|Y}(x|y) \log P_{X|Y}(x|y)$$
$$= \mathsf{E}\left[-\log P_{X|Y}(X|Y) \,\middle|\, Y=y \right].$$

Note that the definition is identical to before apart from that everything is conditioned on the event Y = y

$$0 \le H(X|Y=y) \le \log r;$$
$$H(X|Y=y) = 0 \quad \text{if, and only if,} \quad P(x|y) = 1 \quad \text{for some } x;$$
$$H(X|Y=y) = \log r \quad \text{if, and only if,} \quad P(x|y) = \frac{1}{r} \quad \forall x.$$

Note that the conditional entropy given the event Y = y is a function of y. Since Y is also a RV, we can now average over all possible events Y = y according to the probabilities of each event. This will lead to the averaged.

Mutual Information

Although conditional entropy can tell us when two variables are completely independent, it is not an adequate measure of dependence. A small value for **H(Y| X)** may implies that **X** tells us a great deal about **Y** or that **H**(**Y**) is small to begin with. Thus, we measure dependence using *mutual information*:

I(X,Y) =H(Y)–H(Y|X)

Mutual information is a measure of the reduction of randomness of a variable given knowledge of another variable. Using properties of logarithms, we can derive several equivalent definitions

$$I(X,Y)=H(X)–H(X|\ Y)$$

$$I(X,Y) = H(X)+H(Y)–H(X,Y) = I(Y,X)$$

In addition to the definitions above, it is useful to realize that mutual information is a particular case of the Kullback-Leibler divergence. The KL divergence is defined as:

$$D(p||q) = \int p(x)\log\frac{p(x)}{q(x)}$$

KL divergence measures the difference between two distributions. It is sometimes called the relative entropy. It is always non-negative and zero only when **p=q**; however, it is not a distance because it is not symmetric.

In terms of KL divergence, mutual information is:

$$D(P(X, Y)||P(X)P(Y))) = \int P(X, Y)\log\frac{P(X, Y)}{P(X)P(Y)}$$

In other words, mutual information is a measure of the difference between the joint probability and product of the individual probabilities. These two distributions are equivalent only when **X** and **Y** are independent, and diverge as **X** and **Y** become more dependent.

Source coding

Coding theory is the study of the properties of codes and their respective fitness for specific applications. Codes are used for data compression, cryptography, error-correction, and networking. Codes are studied by various scientific disciplines—such as information theory, electrical engineering, mathematics, linguistics, and computer science—for the purpose of designing efficient and reliable data transmission methods. This typically involves the removal of redundancy and the correction or detection of errors in the transmitted data.

The aim of source coding is to take the source data and make it smaller.

All source models in information theory may be viewed as random process or random sequence models. Let us consider the example of a discrete memory less source (DMS), which is a simple random sequence model.

A DMS is a source whose output is a sequence of letters such that each letter is independently selected from a fixed alphabet consisting of letters; say a_1, a_2 ,

…….a_k. The letters in the source output sequence are assumed to be random and statistically

Independent of each other. A fixed probability assignment for the occurrence of each letter is also assumed. Let us, consider a small example to appreciate the importance of probability assignment of the source letters.

Let us consider a source with four letters a_1, a_2, a_3 and a_4 with $P(a_1)=0.5$, $P(a_2)=0.25$, $P(a_3)= 0.13$, $P(a_4)=0.12$. Let us decide to go for binary coding of these four

Source letters While this can be done in multiple ways, two encoded representations are shown below:

Code Representation#1:

a_1: 00, a_2:01, a_3:10, a_4:11

Code Representation#2:

a_1: 0, a2:10, a_3:001, a_4:110

It is easy to see that in method #1 the probability assignment of a source letter has not been considered and all letters have been represented by two bits each. However in

The second method only a_1 has been encoded in one bit, a_2 in two bits and the remaining two in three bits. It is easy to see that the average number of bits to be used per source letter for the two methods is not the same. ($\bar{a}$ for method #1=2 bits per letter and $\bar{a}$ for method #2 < 2 bits per letter). So, if we consider the issue of encoding a long sequence of

Letters we have to transmit less number of bits following the second method. This is an important aspect of source coding operation in general. At this point, let us note

a) We observe that assignment of small number of bits to more probable letters and assignment of larger number of bits to less probable letters (or symbols) may lead to efficient source encoding scheme.

b) However, one has to take additional care while transmitting the encoded letters. A careful inspection of the binary representation of the symbols in method #2 reveals that it may lead to confusion (at the decoder end) in deciding the end of binary representation of a letter and beginning of the subsequent letter.

So a source-encoding scheme should ensure that

1) The average number of coded bits (or letters in general) required per source letter is as small as possible and
2) The source letters can be fully retrieved from a received encoded sequence.

Shannon-Fano Code

Shannon–Fano coding, named after Claude Elwood Shannon and Robert Fano, is a technique for constructing a prefix code based on a set of symbols and their probabilities. It is suboptimal in the sense that it does not achieve the lowest possible expected codeword length like Huffman coding; however unlike Huffman coding, it does guarantee that all codeword lengths are within one bit of their theoretical ideal $I(x) = -\log P(x)$.

In Shannon–Fano coding, the symbols are arranged in order from most probable to least probable, and then divided into two sets whose total probabilities are as close as possible to being equal. All symbols then have the first digits of their codes assigned; symbols in the first set receive "0" and symbols in the second set receive "1". As long as any sets with more than one member remain, the same process is repeated on those sets, to determine successive digits of their codes. When a set has been reduced to one symbol, of course, this means the symbol's code is complete and will not form the prefix of any other symbol's code.

The algorithm works, and it produces fairly efficient variable-length encodings; when the two smaller sets produced by a partitioning are in fact of equal probability, the one bit of information used to distinguish them is used most efficiently. Unfortunately, Shannon–Fano does not always produce optimal prefix codes.

For this reason, Shannon–Fano is almost never used; Huffman coding is almost as computationally simple and produces prefix codes that always achieve the lowest expected code word length. Shannon–Fano coding is used in the IMPLODE compression method, which is part of the ZIP file format, where it is desired to apply a simple algorithm with high performance and minimum requirements for programming.

Shannon-Fano Algorithm:

A Shannon–Fano tree is built according to a specification designed to define an effective code table. The actual algorithm is simple:

For a given list of symbols, develop a corresponding list of probabilities or frequency counts so that each symbol's relative frequency of occurrence is known.

- Sort the lists of symbols according to frequency, with the most frequently occurring

  Symbols at the left and the least common at the right.
- Divide the list into two parts, with the total frequency counts of the left part being as

  Close to the total of the right as possible.
- The left part of the list is assigned the binary digit 0, and the right part is assigned the digit 1. This means that the codes for the symbols in the first part will all start with 0, and the codes in the second part will all start with 1.

Recursively apply the steps 3 and 4 to each of the two halves, subdividing groups and adding bits to the codes until each symbol has become a corresponding code leaf on the tree.

Example:

The source of information A generates the symbols {A0, A1, A2, A3 and A4} with the corresponding probabilities {0.4, 0.3, 0.15, 0.1 and 0.05}. Encoding the source symbols using binary encoder and Shannon-Fano encoder gives

Source Symbol	**P_i**	**Binary Code**	**Shannon-Fano**
A0	0.4	000	0
A1	0.3	001	10
A2	0.15	010	110
A3	0.1	011	1110
A4	0.05	100	1111
L_{avg}	**H = 2.0087**	**3**	**2.05**

The average length of the Shannon-Fano code is

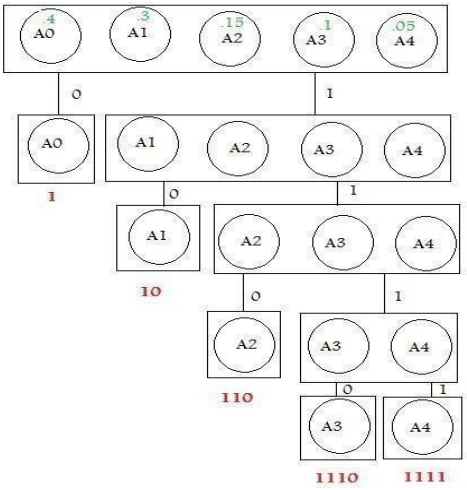
$$\text{Lavg} = \sum_{i=0}^{4} \text{Pi li} = 0.4 * 1 + 0.3 * 2 + 0.15 * 3 + 0.1 * 4 + 0.05 * 4 = 2.05 \text{ bit/symbol}$$

Thus the efficiency of the Shannon-Fano code is

$$\eta = \frac{H}{\text{Lavg}} = \frac{2.0087}{2.05} = 98\%$$

This example demonstrates that the efficiency of the Shannon-Fano encoder is much higher than that of the binary encoder.

Shanon-Fano code is a top-down approach. Constructing the code tree, we get



The Entropy of the source is

$$H = -\sum_{i=0}^{4} \text{Pi } \log_2 \text{Pi} = 2.0087 \text{ bit/symbol}$$

Since we have 5 symbols ($5<8=2^3$), we need 3 bits at least to represent each symbol in binary (fixed-length code). Hence the average length of the binary code is

$$\text{Lavg} = \sum_{i=0}^{4} \text{Pi li} = 3\ (0.4 + 0.3 + 0.15 + 0.1 + 0.05) = 3 \text{ bit/symbol}$$

Thus the efficiency of the binary code is

$$\eta = \frac{H}{\text{Lavg}} = \frac{2.0087}{3} = 67\%$$

Binary Huffman Coding (an optimum variable-length source coding scheme)

In Binary Huffman Coding each source letter is converted into a binary code word. It is a prefix condition code ensuring minimum average length per source letter in bits.

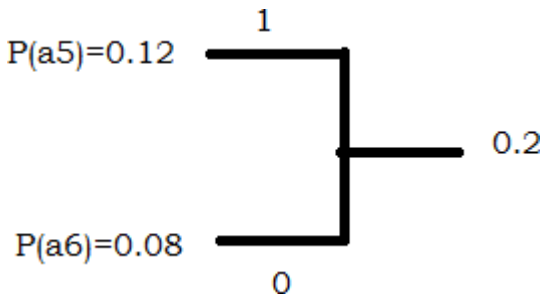
Let the source letters a_1, a_2, …….a_K have probabilities $P(a_1)$, $P(a_2)$,……. $P(a_K)$ and let us assume that $P(a_1) \geq P(a_2) \geq P(a_3) \geq \ldots \geq P(a_K)$.

We now consider a simple example to illustrate the steps for Huffman coding.

Steps to calculate Huffman Coding

Example Let us consider a discrete memory less source with six letters having $P(a_1)=0.3$, $P(a_2)=0.2$, $P(a_3)=0.15$, $P(a_4)=0.15$, $P(a_5)=0.12$ and $P(a_6)=0.08$.

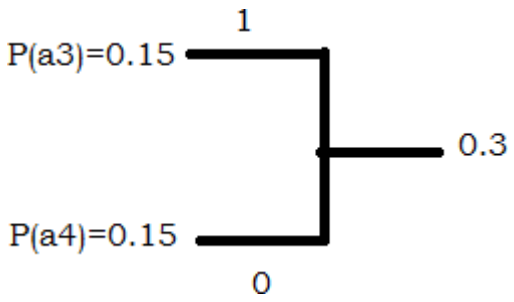
- Arrange the letters in descending order of their probability (here they are arranged).
- Consider the last two probabilities. Tie up the last two probabilities. Assign, say, 0 to the last digit of representation for the least probable letter (a_6) and 1 to the last digit of representation for the second least probable letter (a_5). That is, assign '1' to the upper arm of the tree and '0' to the lower arm.



(3) Now, add the two probabilities and imagine a new letter, say b_1, substituting for a_6 and a_5. So $P(b_1) = 0.2$. Check whether a_4 and b_1 are the least likely letters. If not, reorder the letters as per Step#1 and add the probabilities of two least likely letters. For our example, it leads to:

$P(a_1)=0.3$, $P(a_2)=0.2$, $P(b_1)=0.2$, $P(a_3)=0.15$ and $P(a_4)=0.15$

(4) Now go to Step#2 and start with the reduced ensemble consisting of a_1 , a_2 , a_3 ,



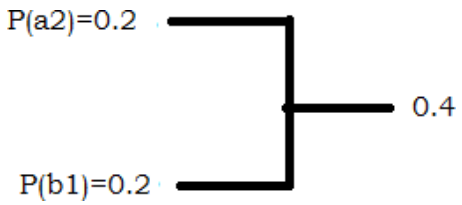
a_4 and b_1. Our example results in:

Here we imagine another letter b_1, with $P(b_2)=0.3$.

Continue till the first digits of the most reduced ensemble of two letters are assigned a '1' and a '0'.

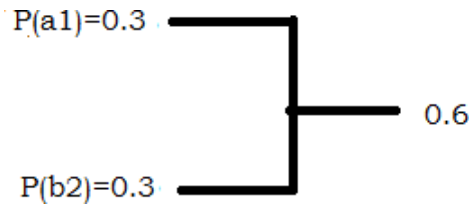
Again go back to the step (2): $P(a_1)=0.3$, $P(b_2)=0.3$, $P(a_2)=0.2$ and $P(b_1)=0.2$.

Now we consider the last two probabilities:

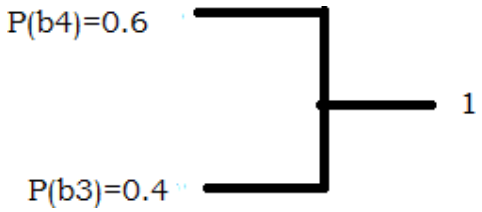


So, $P(b_3)=0.4$. Following Step#2 again, we get, $P(b_3)=0.4$, $P(a_1)=0.3$ and $P(b_2)=0.3$.

Next two probabilities lead to:



With $P(b_4) = 0.6$. Finally we get only two probabilities



6. Now, read the code tree inward, starting from the root, and construct the code words. The first digit of a codeword appears first while reading the code tree inward.

Hence, the final representation is: a_1=11, a_2=01, a_3=101, a_4=100, a_5=001, a_6=000.

A few observations on the preceding example

1. The event with maximum probability has least number of bits

2. Prefix condition is satisfied. No representation of one letter is prefix for other. Prefix condition says that representation of any letter should not be a part of any other letter.

3. Average length/letter (in bits) after coding is

$$= \sum P(a_i) n_i = 2.5 \text{ bits/letter.}$$

4. Note that the entropy of the source is: H(X)=2.465 bits/symbol. Average length per source letter after Huffman coding is a little bit more but close to the source entropy. In fact, the following celebrated theorem due to C. E. Shannon sets the limiting value of average length of code words from a DMS.

Shannon–Hartley theorem

In information theory, the Shannon–Hartley theorem tells the maximum rate at which information can be transmitted over a communications channel of a specified bandwidth in the presence of noise. It is an application of the noisy-channel coding theorem to the archetypal case of a continuous-time analog communications channel subject to Gaussian noise. The theorem establishes Shannon's channel capacity for such a communication link, a

bound on the maximum amount of error-free information per time unit that can be transmitted with a specified bandwidth in the presence of the noise interference, assuming that the signal power is bounded, and that the Gaussian noise process is characterized by a known power or power spectral density.

The law is named after Claude Shannon and Ralph Hartley.

Hartley Shannon Law

The theory behind designing and analyzing channel codes is called Shannon's noisy channel coding theorem. It puts an upper limit on the amount of information you can send in a noisy channel using a perfect channel code. This is given by the following equation:

$$C = B \times \log_2(1 + SNR)$$

where C is the upper bound on the capacity of the channel (bit/s), B is the bandwidth of the channel (Hz) and SNR is the Signal-to-Noise ratio (unit less).

Bandwidth-S/N Tradeoff

The expression of the channel capacity of the Gaussian channel makes intuitive sense:

1. As the bandwidth of the channel increases, it is possible to make faster changes in the information signal, thereby increasing the information rate.

2 As S/N increases, one can increase the information rate while still preventing errors due to noise.

3. For no noise, S/N tends to infinity and an infinite information rate is possible irrespective of bandwidth.

Thus we may trade off bandwidth for SNR. For example, if S/N = 7 and B = 4kHz, then the channel capacity is C = 12 $\times 10^3$ bits/s. If the SNR increases to S/N = 15 and B is decreased to 3kHz, the channel capacity remains the same. However, as B tends to 1, the channel capacity does not become infinite since, with an increase in bandwidth, the noise power also increases. If the noise power spectral density is $\eta/2$, then the total noise power is $N = \eta B$, so the Shannon-Hartley law becomes

$$C = B\log_2\left(1+\frac{S}{\eta B}\right) = \frac{S}{\eta}\left(\frac{\eta B}{S}\right)\log_2\left(1+\frac{S}{\eta B}\right)$$
$$= \frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)^{\eta B/S}.$$

Noting that

$$\lim_{x\to 0}(1+x)^{1/x} = e$$

and identifying x as $x = S/\eta B$, the channel capacity as B increases without bound becomes

$$C_\infty = \lim_{B\to\infty} C = \frac{S}{\eta}\log_2 e = 1.44\frac{S}{\eta}.$$

Linear Block Codes

Introduction

Coding theory is concerned with the transmission of data across noisy channels and the recovery of corrupted messages. It has found widespread applications in electrical engineering, digital communication, mathematics and computer science. The transmission of the data over the channel depends upon two parameters. They are transmitted power and channel bandwidth. The power spectral density of channel noise and these two parameters determine signal to noise power ratio.

The signal to noise power ratio determine the probability of error of the modulation scheme. Errors are introduced in the data when it passes through the channel. The channel noise interferes the signal. The signal power is reduced. For the given signal to noise ratio, the error probability can be reduced further by using coding techniques. The coding techniques also reduce signal to noise power ratio for fixed probability of error.

Principle of block coding

For the block of k message bits, (n-k) parity bits or check bits are added. Hence the total bits at the output of channel encoder are 'n'. Such codes are called (n,k)block codes.Figure illustrates this concept.

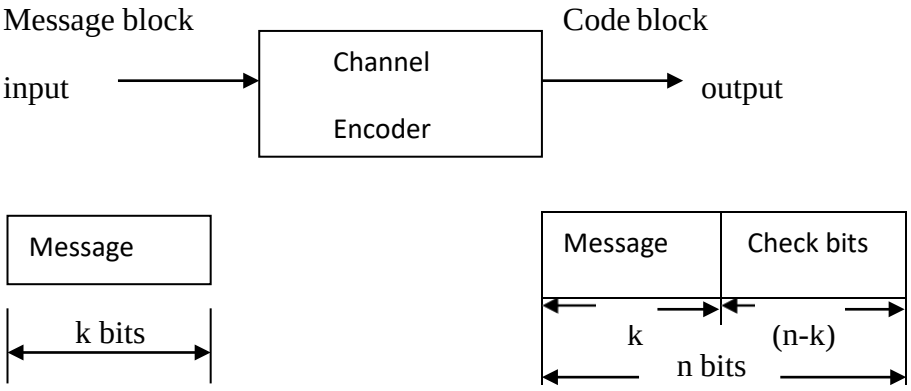


Figure: Functional block diagram of block coder

Types are

Systematic codes:

In the systematic block code, the message bits appear at the beginning of the code word. The message appears first and then check bits are transmitted in a block. This type of code is called systematic code.

Nonsystematic codes:

In the nonsystematic block code it is not possible to identify the message bits and check bits. They are mixed in the block.

Consider the binary codes and all the transmitted digits are binary.

Linear Block Codes

A code is linear if the sum of any two code vectors produces another code vector. This shows that any code vector can be expressed as a linear combination of other code vectors. Consider that the particular code vector consists of $m_1, m_2, m_3, \ldots m_k$ message bits and $c_1, c_2, c_3 \ldots c_q$ check bits. Then this code vector can be written as,

$$X=(m_1, m_2, m_3, \ldots m_k c_1, c_2, c_3 \ldots c_q)$$

Here q=n-k

Whereq are the number of redundant bits added by the encoder.

Code vector can also be written as

$$X=(M/C)$$

Where M= k-bit message vector

C= q-bit check vector

The main aim of linear block code is to generate check bits and this check bits are mainly used for error detection and correction.

Example :

The (7, 4) linear code has the following matrix as a generator matrix

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If u = (1 1 0 1) is the message to be encoded, its corresponding code word would be

$$\begin{aligned} \mathbf{v} &= 1 \cdot \mathbf{g}_0 + 1 \cdot \mathbf{g}_1 + 0 \cdot \mathbf{g}_2 + 1 \cdot \mathbf{g}_3 \\ &= (1\,1\,0\,1\,0\,0\,0) + (0\,1\,1\,0\,1\,0\,0) + (1\,0\,1\,0\,0\,0\,1) \\ &= (0\,0\,0\,1\,1\,0\,1) \end{aligned}$$

A linear systematic (n, k) code is completely specified by ak × n matrix G of the following form

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \mathbf{g}_2 \\ . \\ . \\ . \\ \mathbf{g}_{k-1} \end{bmatrix} = \left[\begin{array}{cccc|cccccccc} p_{00} & p_{01} & . & . & . & p_{0,n-k-1} & 1 & 0 & 0 & . & . & . & 0 \\ p_{10} & p_{11} & . & . & . & p_{1,n-k-1} & 0 & 1 & 0 & . & . & . & 0 \\ p_{20} & p_{21} & . & . & . & p_{2,n-k-1} & 0 & 0 & 1 & . & . & . & 0 \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ & & & & & & & & & & & & \\ p_{k-1,0} & p_{k-1,1} & . & . & . & p_{k-1,n-k-1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(P matrix | $k \times k$ identity matrix)

where $p_{ij} = 0$ or 1

Let u = (u_0, u_1, … , u_{k-1}) be the message to be encoded.The corresponding code word is

$$\begin{aligned} v &= (v_0, v_1, v_2, \ldots, v_{n-1}) \\ &= (u_0, u_1, \ldots, u_{k-1}) \cdot G \end{aligned}$$

The components of **v** are

$$v_{n-k+i} = u_i \qquad \text{for } 0 \le i < k$$

$$v_j = u_0 p_{0j} + u_1 p_{1j} + \cdots + u_{k-1} p_{k-1,j} \qquad \text{for } 0 \le j < n-k$$

The n – k equations given by above equation are called parity-check equations of the code

Example for Codeword

The matrix **G** given by

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let **u** = (u_0, u_1, u_2, u_3) be the message to be encoded and **v** = (v_0, v_1, v_2, v_3, v_4, v_5,v_6) be the corresponding code word

Solution :

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{G} = (u_0, u_1, u_2, u_3) \cdot \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By matrix multiplication, the digits of the code word v can be determined.

$$v_6 = u_3$$
$$v_5 = u_2$$
$$v_4 = u_1$$
$$v_3 = u_0$$
$$v_2 = u_1 + u_2 + u_3$$
$$v_1 = u_0 + u_1 + u_2$$
$$v_0 = u_0 + u_2 + u_3$$

The code word corresponding to the message (1 0 1 1) is (1 0 0 1 0 1 1)

If the generator matrix of an (*n, k*) linear code is in systematic form, the parity-check matrix may take the following form

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{n-k} & \mathbf{P}^T \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & p_{00} & p_{10} & \cdots & p_{k-1,0} \\ 0 & 1 & 0 & \cdots & 0 & p_{01} & p_{11} & \cdots & p_{k-1,1} \\ 0 & 0 & 1 & \cdots & 0 & p_{02} & p_{12} & \cdots & p_{k-1,2} \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & p_{0,n-k-1} & p_{1,n-k-1} & \cdots & p_{k-1,n-k-1} \end{bmatrix}$$

Encoding circuit for a linear systematic (n,k) code is shown below.

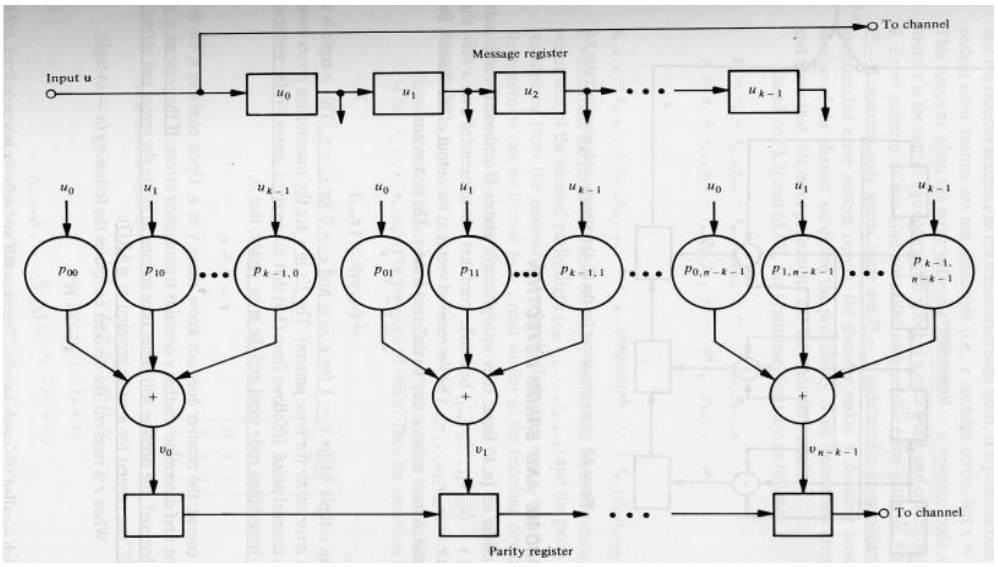


Figure: Encoding Circuit

For the block of k=4 message bits, (n-k) parity bits or check bits are added. Hence the total bits at the output of channel encoder are n=7. The encoding circuit for (7, 4) systematic code is shown below.

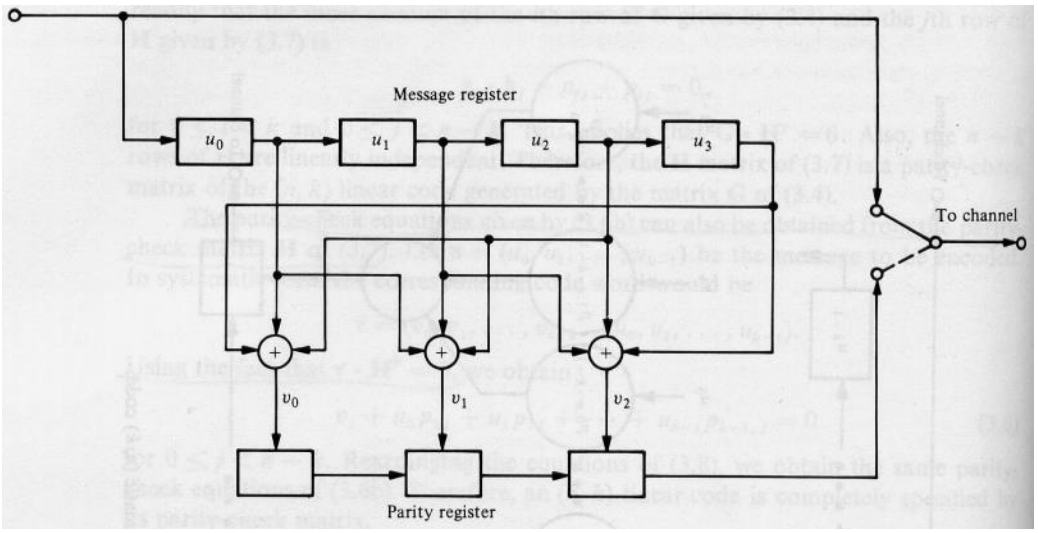
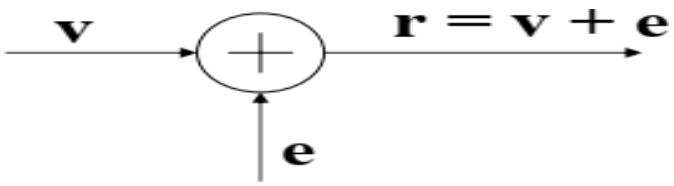


Figure: Encoding Circuit for (7,4) code

Syndrome and Error Detection

Let $v = (v_0, v_1, \ldots, v_{n-1})$ be a code word that was transmitted over a noisy channel. Let $r = (r_0, r_1, \ldots, r_{n-1})$ be the received vector at the outputof the channel



Where

$e = r + v = (e_0, e_1, \ldots, e_{n-1})$ is an n-tuple and the n-tuple 'e' is called the error vector (or error pattern).The condition is

$e_i = 1$ for $r_i \neq v_i$

$e_i = 0$ for $r_i = v_i$

Upon receiving r, the decoder must first determine whether r contains transmission errors. If the presence of errors is detected, the decoder will take actions to locate the errors, correct errors (FEC) and request for a retransmission of v.

When r is received, the decoder computes the following (n – k)-tuple.

$s = r \bullet HT$

$s = (s_0, s_1, \ldots, s_{n-k-1})$

where s is called the syndrome of r.

The syndrome is not a function of the transmitted codeword but a function of error pattern. So we can construct only a matrix of all possible error patterns with corresponding syndrome.

When s = 0, if and only if r is a code word and hence receiver accepts r as the transmitted code word. When $s \neq 0$, if and only if r is not a code word and hence the presence of errors has been detected. When the error pattern e is identical to a nonzero code word (i.e., r contain errors but $s = r \bullet HT = 0$), error patterns of this kind are called undetectable error patterns. Since there are $2^k – 1$ non-zero code words, there are $2^k – 1$ undetectable error patterns. The syndrome digits are as follows:

$s_0 = r_0 + r_{n-k} p_{00} + r_{n-k+1} p_{10} + \cdots + r_{n-1} p_{k-1,0}$

$s1 = r_1 + r_{n-k} p_{01} + r_{n-k+1} p_{11} + \cdots + r_{n-1} p_{k-1,1}$

.

$s_{n-k-1} = r_{n-k-1} + r_{n-k} p_{0,n-k-1} + r_{n-k+1} p_{1,n-k-1} + \cdots + r_{n-1} p_{k-1,n-k-1}$

The syndrome s is the vector sum of the received parity digits $(r_0, r_1, \ldots, r_{n-k-1})$ and the parity-check digits recomputed from the received information digits $(r_{n-k}, r_{n-k+1}, \ldots, r_{n-1})$.

The below figure shows the syndrome circuit for a linear systematic (n, k) code.

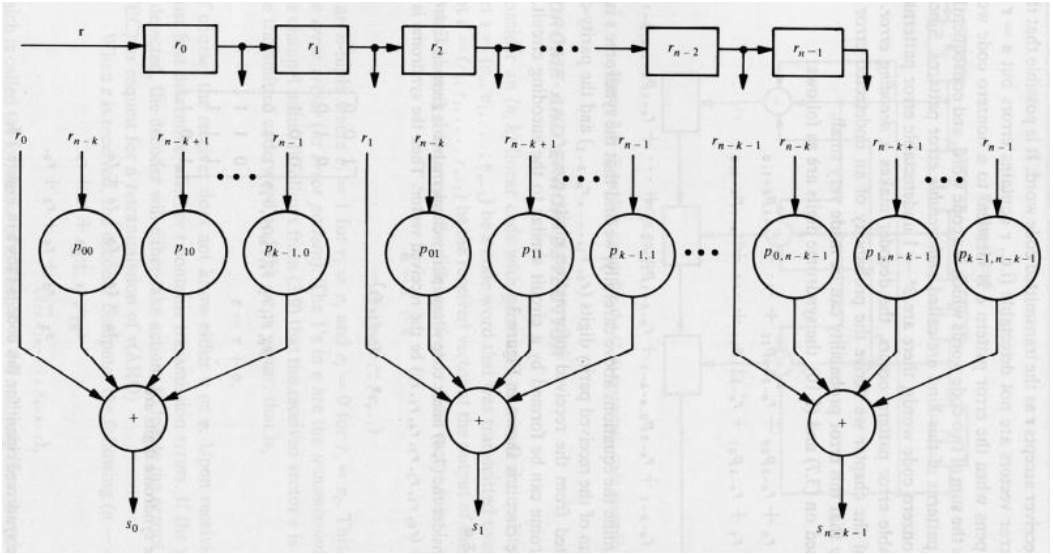


Figure: Syndrome Circuit

Error detection and error correction capabilities of linear block codes:

If the minimum distance of a block code C is d_{min}, any two distinct code vector of C differ in at least d_{min} places. A block code with minimum distance d_{min} is capable of detecting all the error pattern of $d_{min} - 1$ or fewer errors.

However, it cannot detect all the error pattern of d_{min} errors because there exists at least one pair of code vectors that differ in d_{min} places and there is an error pattern of d_{min} errors that will carry one into the other. The random-error-detecting capability of a block code with minimum distance d_{min} is $d_{min} - 1$.

An (n, k) linear code is capable of detecting 2n – 2k error patterns of length n Among the 2n – 1 possible non zero error patterns, there are 2k – 1 error patterns that are identical to the 2k – 1 non zero code words. If any of these 2k – 1 error patterns occurs, it alters the transmitted code word v into another code word w, thus w will be received and its syndrome is zero.

If an error pattern is not identical to a nonzero code word, the received vector r will not be a code word and the syndrome will not be zero.

Hamming Codes:

These codes and their variations have been widely used for error control in digital communication and data storage systems.

For any positive integer $m \geq 3$, there exists a Hamming code with the following parameters:
Code length: $n = 2m - 1$
Number of information symbols: $k = 2m - m - 1$
Number of parity-check symbols: $n - k = m$
Error-correcting capability: $t = 1 (d_{min} = 3)$

The parity-check matrix H of this code consists of all the non zero m-tuple as its columns (2m-1)

In systematic form, the columns of H are arranged in the following form

$$H = [I_m\, Q]$$

where I_m is an $m \times m$ identity matrix

The sub matrix Q consists of $2m - m - 1$ columns which are the m-tuples of weight 2 or more. The columns of Q may be arranged in any order without affecting the distance property and weight distribution of the code.

In systematic form, the generator matrix of the code is

$$G = [Q^T\, I_{2m-m-1}]$$

where Q^T is the transpose of Q and I_{2m-m-1} is an $(2m - m - 1) \times (2m - m - 1)$ identity matrix.

Since the columns of H are nonzero and distinct, no two columns add to zero. Since H consists of all the nonzero m-tuples as its columns, the vector sum of any two columns, say h_i and h_j, must also be a column in H, say $h_l h_i + h_j + h_l = 0$.The minimum distance of a Hamming code is exactly 3.

Using H' as a parity-check matrix, a shortened Hamming code can be obtained with the following parameters :

Code length: $n = 2m - l - 1$

Number of information symbols: $k = 2m - m - l - 1$

Number of parity-check symbols: $n - k = m$

Minimum distance : $d_{min} \geq 3$

When a single error occurs during the transmission of a code vector, the resultant syndrome is nonzero and it contains an odd number of 1's (e x H'T corresponds to a column in H').When double errors occurs, the syndrome is nonzero, but it contains even number of 1's.

Decoding can be accomplished in the following manner:

i) If the syndrome s is zero, we assume that no error occurred

ii) If s is nonzero and it contains odd number of 1's, assume that a single error occurred. The error pattern of a single error that corresponds to s is added to the received vector for error correction.

iii) If s is nonzero and it contains even number of 1's, an uncorrectable error pattern has been detected.

Problems:

1.

The parity check bits of a (8,4) block code are generated by

$$c_0 = m_0 + m_1 + m_3$$
$$c_1 = m_0 + m_1 + m_2$$
$$c_2 = m_0 + m_2 + m_3$$
$$c_3 = m_1 + m_2 + m_3$$

where m_1, m_2, m_3 and m_4 are the message digits.

(a) Find the generator matrix and the parity check matrix for this code.

(b) Find the minimum weight of this code.

(c) Find the error-detecting capabilities of this code.

(d) Show through an example that this code can detect three errors/codeword.

Solution

1(a) $\mathbf{c} = [c_0 \cdots c_8] = [b_0 \cdots b_3 m_0 \cdots m_3] = [m_0 \cdots m_3] \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \vdots \mathbf{I}_4$

Therefore, $\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \vdots \mathbf{I}_4$

and then $\mathbf{H} = \begin{bmatrix} & 1 & 1 & 0 & 1 \\ \mathbf{I}_4 \vdots & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 \end{bmatrix}$

(b)

m	C
0000	0000 0000
0001	1011 0001
0010	0111 0010
0011	1100 0011
0100	1101 0100
0101	0110 0101
0110	1010 0110
0111	0001 0111
1000	1110 1000
1001	0101 1001
1010	1001 1010
1011	0010 1011
1100	0011 1100
1101	1000 1101
1110	0100 1110
1111	1111 1111

Therefore, minimum weight = 4

(c) d_{min} = minimum weight = 4

Therefore, error-detecting capability = $d_{min} - 1 = 3$

(d) Suppose the transmitted code be 00000000 and the received code be 11100000.

$$\mathbf{s} = \mathbf{rH}^{\mathbf{T}} = [11100000]\left[\mathbf{I}_4 \vdots \begin{matrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{matrix}\right]^T = [1110] \neq \mathbf{0}$$

Binary Cyclic codes:

Cyclic codes are the sub class of linear block codes.

Cyclic codes can be in systematic or non systematic form.

Definition:

A linear code is called a cyclic code if every cyclic shift of the code vector produces some other code vector.

Properties of cyclic codes:

(i) Linearity (ii) Cyclic

Linearity: This property states that sum of any two code words is also a valid code word.

$$X_1 + X_2 = X_3$$

Cyclic: Every cyclic shift of valid code vector produces another valid code vector.

Consider an n-bit code vector

$$X = \{x_{n-1}, x_{n-2}, \ldots\ldots\ldots\ldots\ldots x_1, x_0\}$$

Here x_{n-1}, x_{n-2} ….x_1, x_0 represent individual bits of the code vector ‘X’.

If the above code vector is cyclically shifted to left side i.e., One cyclic shift of X gives,

$$X' = \{x_{n-2} \ldots . x_1, x_0, x_{n-1}\}$$

Every bit is shifted to left by one position.

Algebraic Structures of Cyclic Codes:

The code words can be represented by a polynomial. For example consider the n-bit code word $X = \{x_{n-1}, x_{n-2}, \ldots\ldots\ldots\ldots\ldots x_1, x_0\}$.

This code word can be represented by a polynomial of degree less than or equal to (n-1) i.e.,

$$X(p)=x_{n-1}p^{n-1}+x_{n-2}p^{n-2}+\ldots\ldots\ldots\ldots\ldots\ldots +x_1p+x_0$$

Here X(p) is the polynomial of degree (n-1)

p- Arbitrary variable of the polynomial

The power of p represents the positions of the codeword bits i.e.,

p^{n-1} – MSB

p^0 -- LSB

p -- Second bit from LSB side

Polynomial representation due to the following reasons

(i) These are algebraic codes, algebraic operations such as addition, multiplication, division, subtraction etc becomes very simple.

(ii) Positions of the bits are represented with help of powers of p in a polynomial.

Generation of code words in Non-systematic form:

Let M= $\{m_{k-1}, m_{k-2}, \ldots\ldots\ldots\ldots\ldots\ldots m_1, m_0\}$ be 'k' bits of message vector. Then it can be represented by the polynomial as,

$$M(p)=m_{k-1}p^{k-1}+m_{k-2}p^{k-2}+\ldots\ldots\ldots\ldots\ldots\ldots +m_1p+m_0$$

Let X(p) be the code word polynomial

$X(p)=M(p)G(p)$

G(p) is the generating polynomial of degree 'q'

For (n,k) cyclic codes, q=n-k represent the number of parity bits.

The generating polynomial is given as

$$G(p)= p^q+g_{q-1}p^{q-1}+\ldots\ldots\ldots\ldots\ldots +g_1p+1$$

Where $g_{q-1}, g_{q-2}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots g_1$ are the parity bits.

If M_1, M_2, M_3.....etc are the other message vectors, then the corresponding code vectors can be calculated as

$$X_1(p) = M_1(p)\, G(p)$$

$$X_2(p) = M_2(p)\, G(p)$$

$$X_3(p) = M_3(p)\, G(p)$$

Generation of Code vectors in systematic form:

$$X = (k \text{ message bits} : (n-k) \text{ check bits}) = (m_{k-1}, m_{k-2}, \ldots\ldots\ldots\ldots m_1, m_0 : c_{q-1}, c_{q-2}, \ldots\ldots\ldots\ldots c_1, c_0)$$

$$C(p) = c_{q-1}p^{q-1} + c_{q-2}p^{q-2} + \ldots\ldots\ldots\ldots + c_1 p + c_0$$

The check bit polynomial is obtained by

$$C(p) = \text{rem}\left[\frac{p^q M(p)}{G(p)}\right]$$

Generator and Parity Check Matrices of cyclic codes:

Non systematic form of generator matrix:

Since cyclic codes are sub class of linear block codes, generator and parity check matrices can also be defined for cyclic codes.

The generator matrix has the size of k x n.

Let generator polynomial given by equation

$$G(p) = p^q + g_{q-1}p^{q-1} + \ldots\ldots\ldots\ldots + g_1 p + 1$$

Multiply both sides of this polynomial by p^i i.e.,

$$p^i G(p) = p^{i+q} + g_{q-1}p^{i+q-1} \ldots\ldots\ldots\ldots\ldots + g_1 p^{i+1} + p^i \text{ and } i = (k-1), (k-2), \ldots\ldots\ldots\ldots 2, 1, 0$$

Systematic form of generator matrix:

Systematic form of generator matrix is given by

$$G = [I_k : P_{kxq}]_{kxn}$$

The t^{th} row of this matrix will be represented in the polynomial form as follows

$$t^{th} \text{ row of } G = p^{n-t} + R_t(p)$$

Where t= 1, 2, 3 .....k

Lets divide p^{n-t} by a generator matrix G(p). Then we express the result of this division in terms of quotient and remainder i.e.,

$$\frac{p^{n-t}}{G(p)} = Quotient + \frac{Remainder}{G(p)}$$

Here remainder will be a polynomial of degree less than q, since the degree of G(p) is 'q'.

The degree of quotient will depend upon value of t

Lets represent Remainder = $R_t(p)$

Quotient = $Q_t(p)$

$$\frac{p^{n-t}}{G(p)} = Q_t(p) + \frac{R_t(p)}{G(p)}$$

$$p^{n-t} = Q_t(p)G(p) + R_t(p)$$

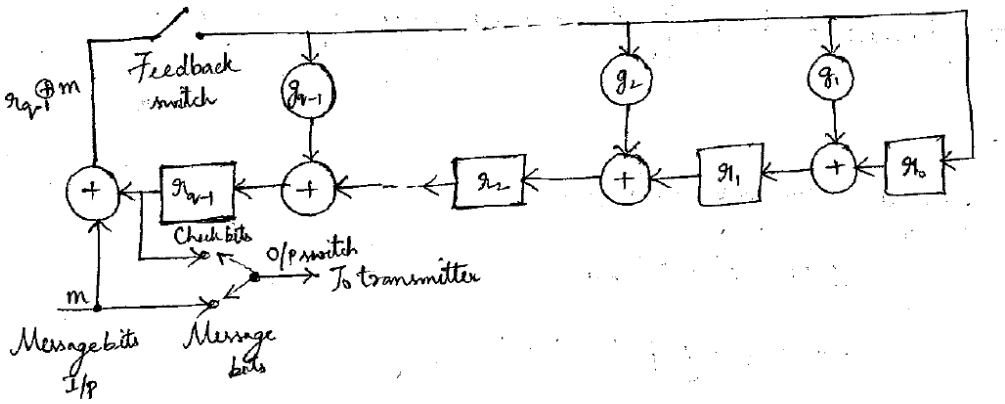
And t= 1,2, ...k

$$p^{n-t} + R_t(p) = Q_t(p)G(p)$$

Represents t^{th} row of systematic generator matrix

Parity check matrix $\mathbf{H = [P^T : I_q]_{qxn}}$

<u>Encoding using an (n-k) Bit Shift Register:</u>



The feedback switch is first closed. The output switch is connected to message input. All the shift registers are initialized to zero state. The 'k' message bits are shifted to the transmitter as well as shifted to the registers.

After the shift of 'k' message bits the registers contain 'q' check bits. The feedback switch is now opened and output switch is connected to check bits position. With the every shift, the check bits are then shifted to the transmitter.

The block diagram performs the division operation and generates the remainder. Remainder is stored in the shift register after all message bits are shifted out.

Syndrome Decoding, Error Detection and Error Correction:

In cyclic codes also during transmission some errors may occur. Syndrome decoding can be used to correct those errors.

Lets represent the received code vector by Y.

If 'E' represents the error vector then the correct code vector can be obtained as

X=Y+E or Y=X+E

In the polynomial form we can write above equation as

$$Y(p) = X(p)+E(p)$$

$$X(p) = M(p)G(p)$$

$$Y(p)= M(p)G(p) + E(p)$$

$$\frac{Y(p)}{G(p)} = Quotient + \frac{Remainder}{G(p)}$$

If Y(p)=X(p)

$$\frac{X(p)}{G(p)} = Quotient + \frac{Remainder}{G(p)}$$

$$\frac{Y(p)}{G(p)} = Q(p) + \frac{R(p)}{G(p)}$$

$$Y(p)=Q(p)G(p) + R(p)$$

Clearly R(p) will be the polynomial of degree less than or equal to q-1

$$Y(p) = Q(p) G(p) + R(p)$$

$$M(p)G(p)+E(p)=Q(p)G(p)+R(p)$$

$$E(p)=M(p)G(p)+Q(p)G(p)+ R(p)$$

$$E(p)=[M(p)+Q(p)]G(p)+R(p)$$

This equation shows that for a fixed message vector and generator polynomial, an error pattern or error vector 'E' depends on remainder R.

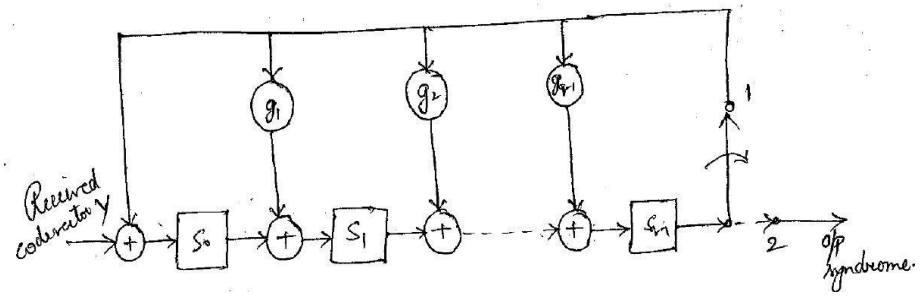
For every remainder 'R' there will be specific error vector. Therefore we can call the remainder vector 'R' as syndrome vector 'S', or R(p)=S(p). Therefore

$$\frac{Y(p)}{G(p)} = Q(p) + \frac{S(p)}{G(p)}$$

Thus Syndrome vector is obtained by dividing received vector Y (p) by G (p) i.e.,

$$S(p) = rem[\frac{Y(p)}{G(p)}]$$

Block Diagram of Syndrome Calculator:



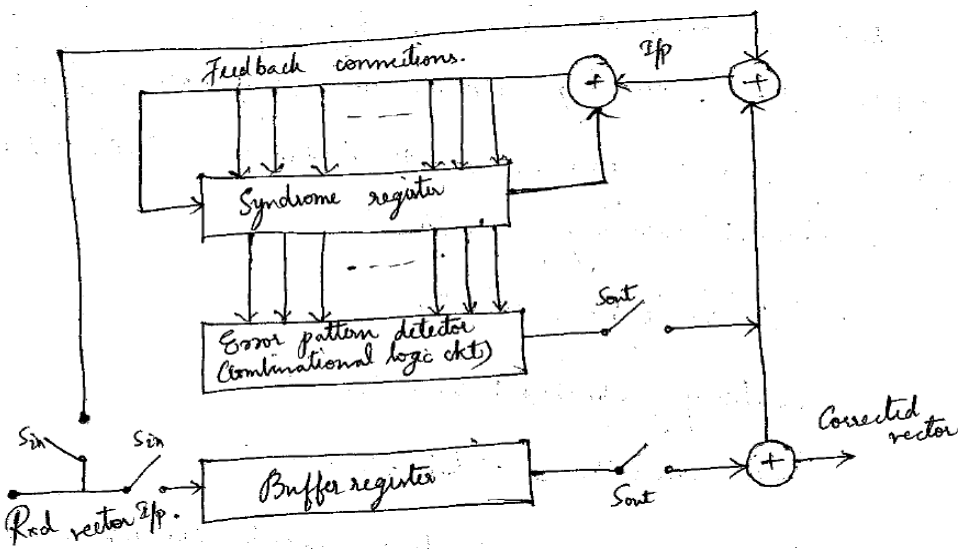
There are 'q' stage shift register to generate 'q' bit syndrome vector. Initially all the shift register contents are zero & the switch is closed in position 1.

The received vector Y is shifted bit by bit into the shift register. The contents of flip flops keep changing according to input bits of Y and values of g1,g2 etc.

After all the bits of Y are shifted, the 'q' flip flops of shift register contain the q bit syndrome vector. The switch is then closed to position 2 & clocks are applied to shift register. The output is a syndrome vector S= (S_{q-1}, S_{q-2} ….S_1, S_0)

Decoder of Cyclic Codes:

Once the syndrome is calculated, then an error pattern is detected for that particular syndrome. When the error vector is added to the received code vector Y, then it gives corrected code vector at the output.



The switch named Sout is opened and Sin is closed. The bits of the received vector Y are shifted into the buffer register as well as they are shifted in to the syndrome calculator. When all the n bits of the received vector Y are shifted into the buffer register and Syndrome calculator the syndrome register holds a syndrome vector.

Syndrome vector is given to the error pattern detector. A particular syndrome detects a specific error pattern.

Sin is opened and Sout is closed. Shifts are then applied to the flip flop of buffer registers, error register, and syndrome register.

The error pattern is then added bit by bit to the received vector. The output is the corrected error free vector.

Convolution codes

Definition of Convolutional Coding

A convolutional coding is done by combining the fixed number of input bits. The input bits are stored in the fixed length shift register and they are combined with the help of mod-2 adders. This operation is equivalent to binary convolution and hence it is called *convolutional coding*. This concept is illustrated with the help of simple

The output switch first samples x_1 and then x_2. The shift register then shifts contents of m_1 to m_2 and contents of m to m_1. Next input bit is then taken and stored in m. Again x_1 and x_2 are generated according to this new combination of m, m_1 and m_2 (equation 4.4.1 and equation 4.4.2). The output switch then samples x_1 then x_2. Thus the output bit stream for successive input bits will be,

$$x_1 = m \oplus m_1 \oplus m_2 \quad \ldots (4.4.1)$$

and

$$x_2 = m \oplus m_2 \quad \ldots (4.4.2)$$

$$X = x_1 x_2 x_1 x_2 x_1 x_2 \ldots \text{ and so on} \quad \ldots (4.4.3)$$

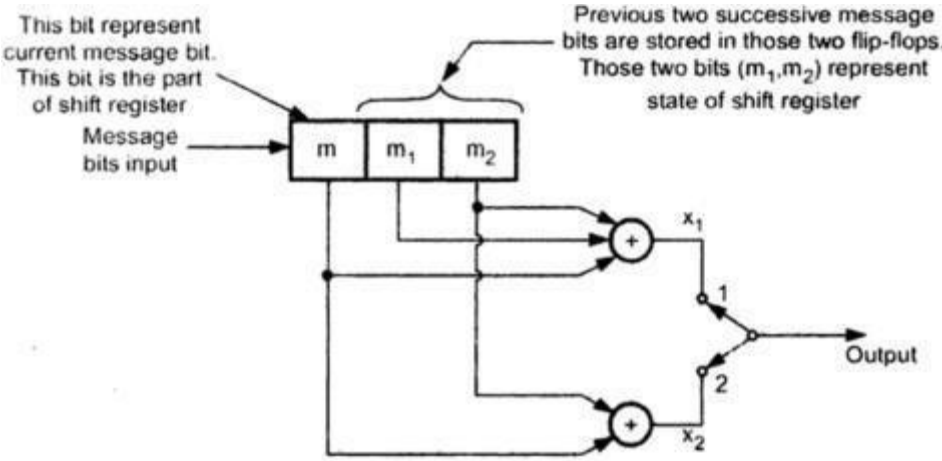


Fig. 4.4.1 Convolutional encoder with k = 3, k = 1 and n = 2

Operation :

Whenever the message bit is shifted to position 'm', the new values of x_1 and x_2 are generated depending upon m, m_1 and m_2. m_1 and m_2 store the previous two message bits. The current bit is present in m. Thus we can write,

Here note that for every input message bit two encoded output bits x_1 and x_2 are transmitted. In other words, for a single message bit, the encoded code word is two bits i.e. for this convolutional encoder,

Number of message bits, k = 1

Number of encoded output bits for one message bit, n = 2

4.4.1.1 Code Rate of Convolutional Encoder

The code rate of this encoder is,

$$r = \frac{k}{n} = \frac{1}{2} \qquad \ldots (4.4.4)$$

In the encoder of Fig. 4.4.1, observe that whenever a particular message bit enters a shift register, it remains in the shift register for three shifts i.e.,

First shift → Message bit is entered in position 'm'.

Second shift → Message bit is shifted in position m_1.

Third shift → Message bit is shifted in position m_2.

And at the fourth shift the message bit is discarded or simply lost by overwriting. We know that x_1 and x_2 are combinations of m, m_1, m_2. Since a single message bit remains in m during first shift, in m_1 during second shift and in m_2 during third shift; it influences output x_1 and x_2 for 'three' successive shifts.

4.4.1.2 Constraint Length (K)

The constraint length of a convolution code is defined as the number of shifts over which a single message bit can influence the encoder output. It is expressed in terms of message bits.

For the encoder of Fig. 4.4.1 constraint length K = 3 bits. This is because in this encoder, a single message bit influences encoder output for three successive shifts. At the fourth shift, the message bit is lost and it has no effect on the output.

4.4.1.3 Dimension of the Code

The dimension of the code is given by n and k. We know that 'k' is the number of message bits taken at a time by the encoder. And 'n' is the encoded output bits for one message bits. Hence the dimension of the code is (n, k). And such encoder is called (n, k) convolutional encoder. For example, the encoder of Fig. 4.4.1 has the dimension of (2, 1).

4.4.2 Time Domain Approach to Analysis of Convolutional Encoder

Let the sequence $\{g_0^{(1)}, g_1^{(1)}, g_2^{(1)} \ldots\ldots g_m^{(1)}\}$ denote the impulse response of the adder which generates x_1 in Fig. 4.4.1 Similarly, Let the sequence $\{g_0^{(2)}, g_1^{(2)}, g_2^{(2)} \ldots\ldots g_m^{(2)}\}$ denote the impulse response of the adder which generates x_2 in Fig. 4.4.1. These impulse responses are also called *generator sequences* of the code.

Let the incoming message sequence be $\{m_0, m_1, m_2 \ldots\ldots\}$. The encoder generates the two output sequences x_1 and x_2. These are obtained by convolving the generator sequences with the message sequence. Hence the name convolutional code is given. The sequence x_1 is given as,

$$x_1 = x_i^{(1)} = \sum_{l=0}^{M} g_l^{(1)} m_{i-l} \qquad i = 0, 1, 2 \ldots\ldots \qquad \ldots (4.4.6)$$

Here $m_{i-l} = 0$ for all $l > i$. Similarly the sequence x_2 is given as,

$$x_2 = x_i^{(2)} = \sum_{l=0}^{M} g_l^{(2)} m_{i-l} \qquad i = 0, 1, 2 \ldots\ldots \qquad \ldots (4.4.7)$$

Note : All additions in above equations are as per mod-2 addition rules.

As shown in the Fig. 4.4.1, the two sequences x_1 and x_2 are multiplexed by the switch. Hence the output sequence is given as,

$$\{x_i\} = \left\{ x_0^{(1)}\ x_0^{(2)}\ x_1^{(1)}\ x_1^{(2)}\ x_2^{(1)}\ x_2^{(2)}\ x_3^{(1)}\ x_3^{(2)} \ldots\ldots \right\} \qquad \ldots (4.4.8)$$

$$v_1 = x_i^{(1)} = \left\{ x_0^{(1)}\ x_1^{(1)}\ x_2^{(1)}\ x_3^{(1)} \ldots\ldots \right\}$$

$$v_2 = x_i^{(2)} = \left\{ x_0^{(2)}\ x_1^{(2)}\ x_2^{(2)}\ x_3^{(2)} \ldots\ldots \right\}$$

Observe that bits from above two sequences are multiplexed in equation (4.4.8) The sequence $\{x_i\}$ is the output of the convolutional encoder.

Transform Domain Approach to Analysis of Convolutional Encoder

In the previous section we observed that the convolution of generating sequence and message sequence takes place. These calculations can be simplified by applying the transformations to the sequences. Let the impulse responses be represented by polynomials. i.e.,

$$g^{(1)}(p) = g_0^{(1)} + g_1^{(1)}p + g_2^{(1)}p^2 + \ldots\ldots + g_M^{(1)}p^M \quad \ldots (4.4.13)$$

$$g^{(2)}(p) = g_0^{(2)} + g_1^{(2)}p + g_2^{(2)}p^2 + \ldots\ldots + g_M^{(2)}p^M \quad \ldots (4.4.14)$$

Thus the polynomials can be written for other generating sequences. The variable 'p' is unit delay operator in above equations. It represents the time delay of the bits in impulse response.

Similarly we can write the polynomial for message polynomial i.e.,

$$m(p) = m_0 + m_1p + m_2p^2 + \ldots\ldots + m_{L-1}p^{L-1} \quad \ldots (4.4.15)$$

Here L is the length of the message sequence. The convolution sums are converted to polynomial multiplications in the transform domain. i.e.,

$$\boxed{\begin{aligned} x^{(1)}(p) &= g^{(1)}(p) \cdot m(p) \\ x^{(2)}(p) &= g^{(2)}(p) \cdot m(p) \end{aligned}} \quad \ldots (4.4.16)$$

The above equations are the output polynomials of sequences $x_i^{(1)}$ and $x_i^{(2)}$.

Code Tree, Trellis and State Diagram for a Convolution Encoder

Now let's study the operation of the convolutional encoder with the help of code tree, trellis and state diagram. Consider again the convolutional encoder of Fig. 4.4.1. It is reproduced below for convenience.

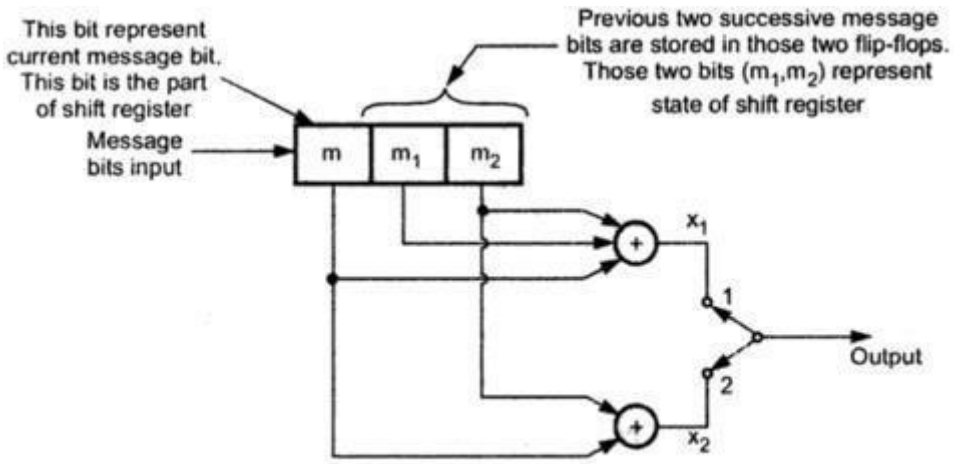


Fig. 4.4.4 Convolutional encoder with k = 1 and n = 2

States of the Encoder

In Fig. 4.4.4 the previous two successive message bits m_1 and m_2 represents state. The input message bit m affects the 'state' of the encoder as well as outputs x_1 and x_2 during that state. Whenever new message bit is shifted to 'm', the contents of m_1 and m_2 define new state. And outputs x_1 and x_2 are also changed according to new state m_1, m_2 and message bit m. Let's define these states as shown in Table 4.4.1.

Let the initial values of bits stored in m_1 and m_2 be zero. That is $m_1 m_2 = 00$ initially and the encoder is in state 'a'.

m_2	m_1	State of encoder
0	0	a
0	1	b
1	0	c
1	1	d

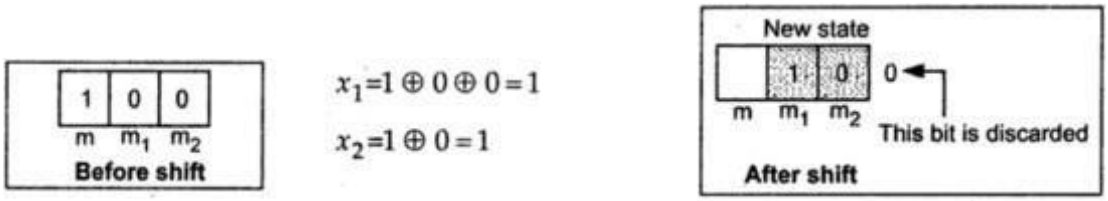
Table 4.4.1 States of the encoder of Fig. 4.4.4

Development of the Code Tree

Let us consider the development of code free for the message sequence m = 110. Assume that $m_1 m_2 = 00$ initially.

1) When $m = 1$ i.e. first bit

The first message input is m = 1. With this input x_1 and x_2 will be calculated as



The values of x_1x_2 = 11 are transmitted to the output and register contents are shifted to right by one bit position as shown.

Thus the new state of encoder is $m_2 m_1 = 01$ or *'b'* and output transmitted are $x_1x_2 = 11$. This shows that if encoder is in state 'a' and if input is m = 1 then the next state is *'b'* and outputs are $x_1x_2 = 11$. The first row of Table 4.4.2 illustrates this operation.

The last column of this table shows the code tree diagram. The code tree diagram starts at node or state *'a'*. The diagram is reproduced as shown in Fig. 4.4.5.

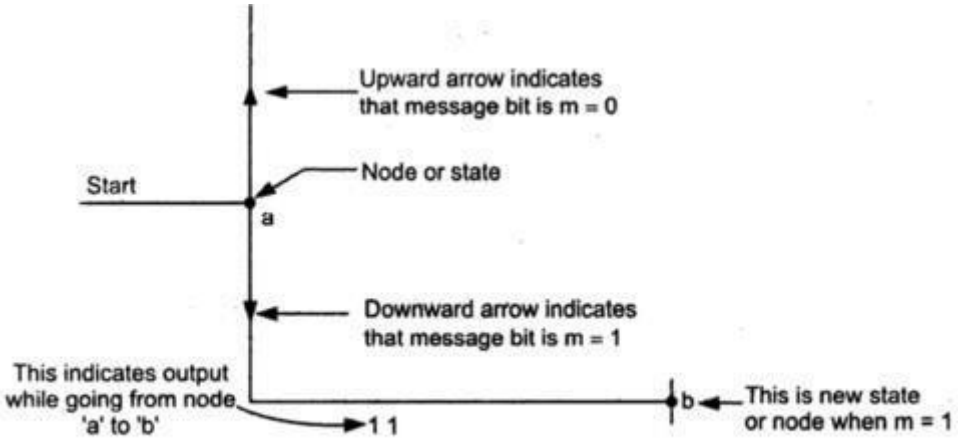
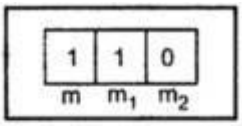


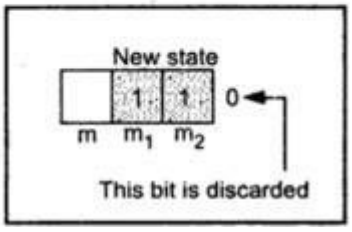
Fig. 4.4.5 Code tree from node *'a'* to *'b'*

Observe that if m = 1 we go downward from node *'a'*. Otherwise if m = 0, we go upward from node *'a'*. It can be verified that if m = 0 then next node (state) is 'a' only. Since m = 1 here we go downwards toward node b and output is 11 in this node (or state).

2) When $m = 1$ i.e. second bit

Now let the second message bit be 1. The contents of shift register with this input will be as shown below.





$x_1 = 1 \oplus 1 \oplus 0 = 0$

$x_2 = 1 \oplus 0 = 1$

These values of $x_1x_2 = 01$ are then transmitted to output and register contents are shifted to right by one bit. The next state formed is as shown.

Thus the new state of the encoder is $m_2m_1 = 11$ or *'d'* and the outputs transmitted are $x_1x_2 = 01$. Thus the encoder goes from state 'b' to state *'d'* if input is '1' and transmitted output $x_1x_2 = 01$. This operation is illustrated by Table 4.4.2 in second row. The last column of the table shows the code tree for those first and second input bits.

Example 4.4.8 : *Determine the state diagram for the convolutional encoder shown in Fig. 4.4.32. Draw the trellis diagram through the first set of steady state transitions. On the second trellis diagram, show the termination of trellis to all zero state.*

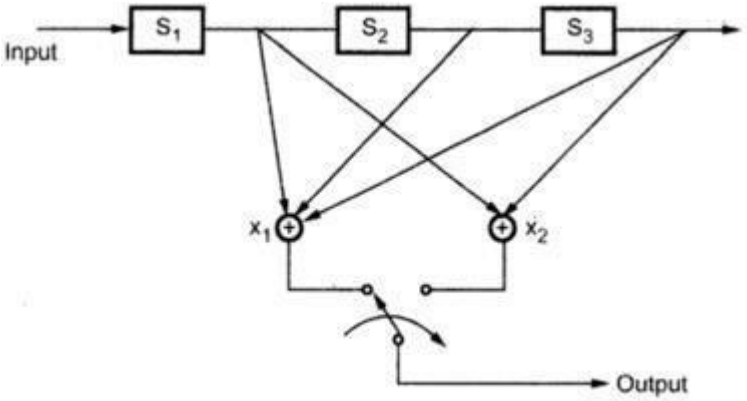


Fig. 4.4.32 Convolutional encoder of example 4.4.8

Sol. : (i) To determine dimension of the code :

For every message bit $(k = 1)$, two output bits $(n = 2)$ are generated. Hence this is rate $\frac{1}{2}$ code. Since there are three stages in the shift register, every message bit will affect output for three successive shifts. Hence constraint length, $K = 3$. Thus,

$$k = 1, \quad n = 2 \text{ and } K = 3$$

ii) To obtain the state diagram :

First, let us define the states of the encoder.

$s_3 s_2 = 0\ 0$, state 'a'

$s_3 s_2 = 0\ 1$, state 'b'

$s_3 s_2 = 1\ 0$, state 'c'

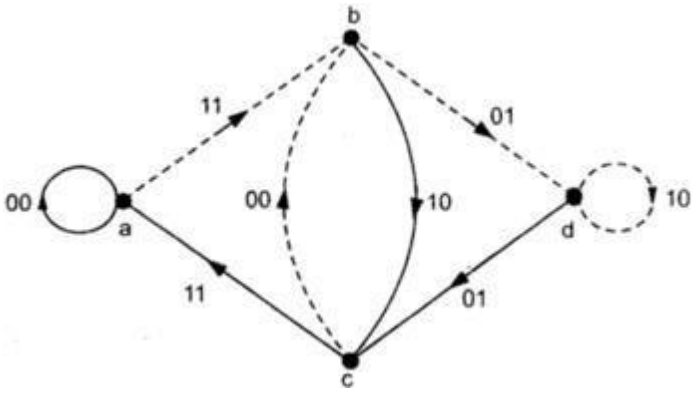
$s_3 s_2 = 1\ 1$, state 'd'

A table is prepared that lists state transitions, message input and outputs. The table is as follows :

Sr. No.	Current state $s_3\ s_2$	Input s_1	Outputs $x_1 = s_1 \oplus s_2 \oplus s_3$ $x_2 = s_1 \oplus s_3$		Next state $s_2\ s_1$
1	a = 0 0	0	0	0	0 0, i.e. a
		1	1	1	0 1, i.e. b
2	b = 0 1	0	1	0	1 0, i.e. c
		1	0	1	1 1, i.e. d
3	c = 1 0	0	1	1	0 0, i.e. a
		1	0	0	0 1, i.e. b
4	d = 1 1	0	0	1	1 0, i.e. c
		1	1	0	1 1, i.e. d

Table 4.4.8 : State transition table

Based on above table, the state diagram can be prepared easily. It is shown below in Fig. 4.4.33.



iii) To obtain trellis diagram for steady state :

From Table 4.4.9, the code trellis diagram can be prepared. It is steady state diagram. It is shown below.

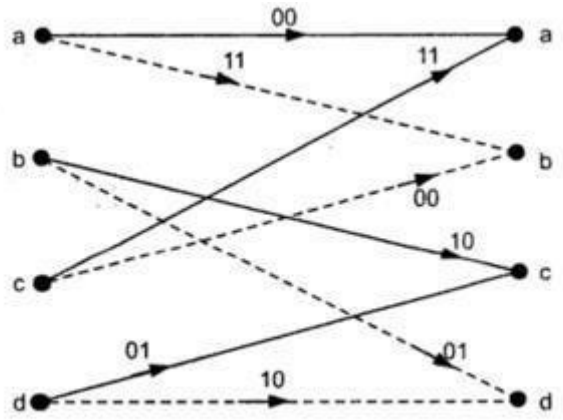


Fig. 4.4.34 Code trellis diagram for steady state

Decoding methods of Convolution code:

1.Veterbi decoding

2.Sequential decoding

3.Feedback decoding

Veterbi algorithm for decoding of convolution codes(maximam likelihood decoding):

Let represent the received signal by y.

Convolutional encoding operates continuously on input data

Hence there areno code vectorsand blocks such as.

Metric:it is the discrepancybetwen the received signal y and the decoding signal at particular node .this metric can be added over few nodes a particular path

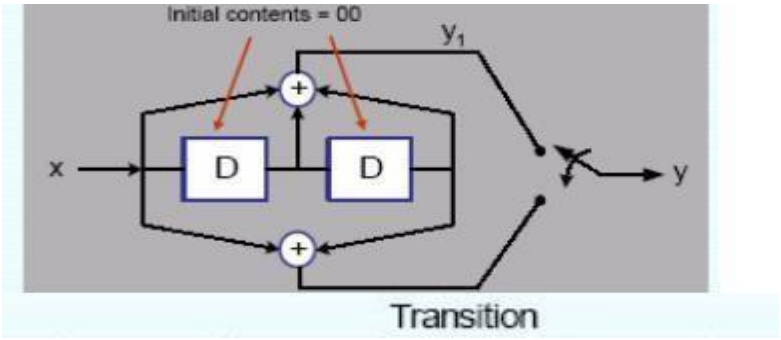
Surviving path: this is the path of the decoded signalwith minimum metric

In veterbi decoding ametric isassigned to each surviving path

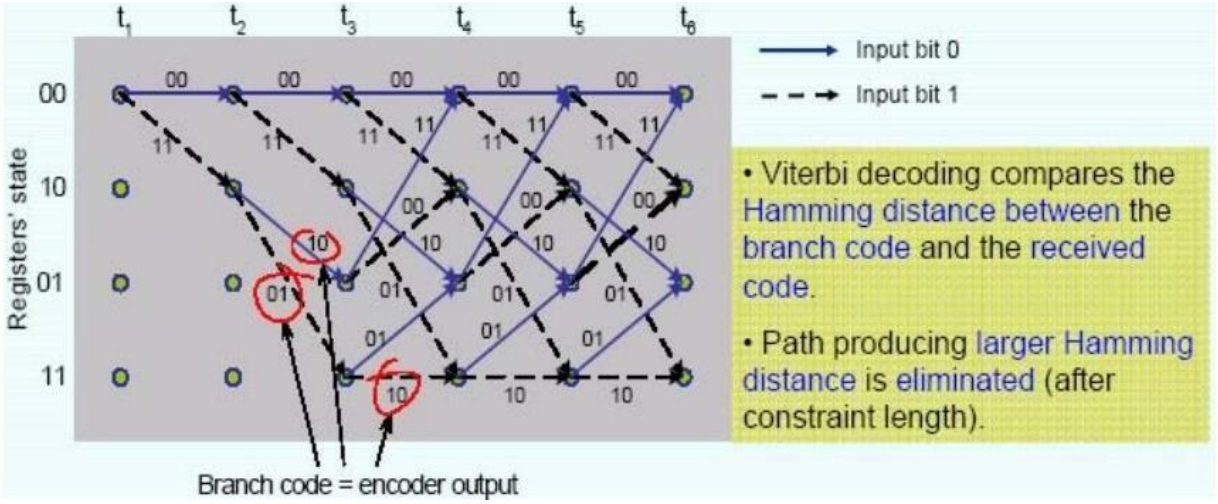
Metric of the particular is obtained by adding individual metric on the nodes along that path.

Y is decoded as the surviving path with smallest metric.

Example:



Exe:



Input data : m =1 1 0 1 1
Codeword : X = 11 01 01 00 01
Received code : Z = 11 01 01 10 01